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Conditional Entropy and the Rokhlin Metric on an Orthomodular Lattice with Bayessian State

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Abstract The present paper deals with the study of conditional entropy and its properties in a quantum space (L, s), where L is an orthomodular lattice and s is a Bayessian state on L. First, we obtained a pseudo-metric on the family of all partitions of the couple (B, s), where B is a Boolean algebra and s is a state on B. This pseudo-metric turns out to be a metric (called the Rokhlin metric) by using a new notion of s-refinement and by identifying those partitions of (B, s) which are s-equivalent. The present theory has then been extended to the quantum space (L, s), where L is an orthomodular lattice and s is a Bayessian state on L. Applying the theory of commutators and Bell inequalities, it is shown that the couple (L, s) can be equivalently replaced by a couple (B, s_0), where B is a Boolean algebra and s_0 is a state on B.

Keywords Boolean algebra · Orthomodular lattice · State · Partition · Entropy · Rokhlin metric

1 Introduction

Consequent to the introduction of a new model for quantum mechanics by Riečan and Dvurečenskij [24], several authors have made their contributions in this direction which can be seen in [5, 7, 8, 11, 14, 15, 24, 26, 27]. Orthomodular posets and orthomodular lattices play a fundamental role in the quantum logic theory [3, 10]. Quantum logic is not modular, but satisfies a weaker form of modularity, (called orthomodularity) which holds for those elements which are orthogonal. Effect algebras or equivalently D-posets [1, 6, 9, 13] can be considered as a generalized form of quantum logic, which for some reasons are also referred to as unsharp quantum logic. Using the notion of a state (or measure) one can introduce the

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concept of entropy of partitions in the theory of Boolean algebras, which is a useful tool in the study of the isomorphism of dynamical systems [4, 6, 18] and has been applied in many others structures. Recently in 2003, Riečan [23] constructed the entropy of a dynamical system on an arbitrary MV-algebra, while Yuan [29] tried to introduce the entropy of partitions on quantum logic (or a σ-orthomodular lattice).

In the present paper, we put forward the concepts of entropy and conditional entropy in a quantum space, which is defined as a couple (L, s), where L is an orthomodular lattice and s is a state on L. In particular, we will be concerned with the case when the state s has the Bayes' property (or is a Bayessian state), which was introduced recently in [29]. It turns out that a Bayessian state annihilates all (upper) commutators in L. Applying the theory of commutators and boolean quotients in orthomodular lattices [3, 16, 17, 19] and Bell inequalities [21, 22], the couple (L, s) can be equivalently replaced by a couple (B, s_0), where B is a Boolean algebra and s_0 is a state on B. Notice that every state on a Boolean algebra is Bayessian. Therefore, we begin our study with a couple (B, s), where B is a Boolean algebra and s is a state on B, and extend the concept of the Rokhlin metric based on the conditional entropy. In Sect. 3, notions of a partition $\mathcal{A}$ of (B, s), common refinement of partitions, entropy $H_s(\mathcal{A})$ of $\mathcal{A}$, conditional entropy $H_s(\mathcal{A}|\mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are partitions of (B, s), are introduced and studied; some results are proved which are necessary for the study made in the subsequent sections, where a pseudo-metric on the couple (B, s) is obtained. The concept of s-refinement of partitions is given which gives rise to an equivalence relation on the family $\mathfrak{P}_s$ of all partitions of (B, s). Finally, we obtain the Rokhlin metric on the resulting quotient space. Then we extend the present study to the quantum space (L, s), where L is an orthomodular lattice and s is a Bayessian state.

2 Preliminaries (cf. [9, 28])

An *orthomodular poset* (*OMP*) is a bounded poset $L = (L, \leq, \vee, \wedge, 0, 1)$ which contains smallest element 0 and the greatest element 1, with a unary operation $' : L \to L$ such that the following conditions are satisfied for all $a, b, c \in L$:

(1) $a \leq b \Rightarrow b' \leq a'$;
(2) $(a')' = a$;
(3) $a \leq b' \Rightarrow a \vee b$ exists in L;
(4) (*Orthomodular law*) $a \leq b \Rightarrow \exists\, c \in L$ such that $c \leq a'$ and $a \vee c = b$.

As a consequence of the orthomodular law, we get $a \vee a' = 1$. Two elements $a, b \in L$ are called *orthogonal* if $a \leq b'$ denoted by $a \perp b$. The following properties hold in an OMP L, for every $a, b \in L$:

(1) $0' = 1$ and $1' = 0$;
(2) if $a \vee b \in L$, then $(a \vee b)' = a' \wedge b'$;
(3) $a \wedge a' = 0$;
(4) if $a \wedge b \in L$, then $(a \wedge b)' = a' \vee b'$;
(5) if $a \leq b$, then $b = a \vee (a \vee b')'$.

Property (3) is a consequence of property (2), and property (5) is equivalent to the orthomodular law. An *orthomodular lattice* (*OML*) is an OMP that is also a lattice. A *quantum logic* is a σ-orthomodular lattice (σ-OML), i.e. an orthomodular lattice with condition (3) replaced by: given any countable sequence $\{a_i\}_{i=1}^{\infty} \subseteq L$, $a_i \leq a_j'$, $\forall i \neq j$, $\bigvee_{i=1}^{\infty} a_i$ exists in L. As known, a typical example of an OML is the lattice of all closed subspaces of a Hilbert space or a Boolean algebra. An OML L is *Boolean* (i.e. it is a *Boolean algebra*) exactly if it is distributive. For an OML L, the following are equivalent:

(1) L is a Boolean algebra.
(2) L is distributive.
(3) All elements of L commute with each other.

The orthomodular law is a kind of distributivity: for $a \le b$, we have $a \vee (a' \wedge b) = b = 1 \wedge b = (a \vee a') \wedge (a \vee b)$. Also recall that if an OML L satisfies: $a \wedge b = 0 \Rightarrow a \le b'$, then L is a Boolean algebra.

A *state* on an OML L is a map $s : L \to [0, 1]$ satisfying:

(1) $s(1) = 1$;
(2) for $a, b \in L$ with $a \perp b$, $s(a \vee b) = s(a) + s(b)$.

It may be observed that $s(0) = 0$, s is monotone and $s(a') = 1 - s(a)$, $a \in L$. Further, a state s on L is called *subadditive* if, in addition, it fulfills the following condition: $s(a \vee b) \le s(a) + s(b)$, for any $a, b \in L$. A state s on L is called a *modular state* if $s(a \vee b) = s(a) + s(b)$, provided $a \wedge b = 0$, or equivalently $s(a \vee b) + s(a \wedge b) = s(a) + s(b)$, for any $a, b \in L$. Evidently, every modular state on an OML L is a subadditive state on L. Indeed, every subadditive state on L is a modular state. An OML L is called *unital* with respect to subadditive states on L, if for any non-zero $a \in L$, there is a subadditive state s on L such that $s(a) = 1$. An OML is a Boolean algebra if and only if it is unital with respect to subadditive states. Suppose that for any non-zero element $a \in L$ there is such a modular state s on L that $s(a) = 1$, then L is a Boolean algebra. It may be noted that, for any Boolean algebra B, every state s on B is a modular state [20], and has the following property: For a given $a \in L$,

$$s(a \wedge b) = s(b), \quad \forall b \in L \quad \Longleftrightarrow \quad s(a) = 1. \tag{2.1}$$

Let $a, b \in B$ and s be a state on a Boolean algebra B. Then the *conditional state* is given by

$$s(a|b) = \begin{cases} \dfrac{s(a \wedge b)}{s(b)}, & \text{if } s(b) > 0, \\ 0, & \text{if } s(b) = 0. \end{cases}$$

3 Partitions of (B, s) and Entropy

We begin our study with a couple (B, s), where $B = (B, \le, \vee, \wedge, 0, 1)$ is a Boolean algebra and s is a state on B. Denote by $\mathbb{R}$ the set of all real numbers, and by $\mathbb{N}$ the set of all positive integers.

Definition 3.1 A (finite) system $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ of elements of a Boolean algebra B is said to be a $\vee$-orthogonal system if $(\bigvee_{i=1}^{k}, a_i) \perp a_{k+1}$ for $k = 1, 2, 3, \ldots, n - 1$.

For any $\vee$-orthogonal system $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ of a Boolean algebra B and any state s on B, $s(\bigvee_{i=1}^{n} a_i) = \sum_{i=1}^{n} s(a_i)$.

The system $\mathcal{A}$ is said to be a partition of B corresponding to a state s defined on B if

(1) $\mathcal{A}$ is a $\vee$-orthogonal system;
(2) $s(\bigvee_{i=1}^{n} a_i) = 1$.

By a partition $\mathcal{A}$ of a couple (B, s) we mean that $\mathcal{A}$ is a partition of B corresponding to the state s.

Let $\mathcal{B} = \{b_1, b_2, \dots, b_m\}$ be a partition of a couple (B, s) and $a \in B$. Then by (2.1)

$$\sum_{j=1}^{m} s(a \wedge b_j) = s(a). \tag{3.1}$$

Let $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$ and $\mathcal{B} = \{b_1, b_2, \dots, b_m\}$ be two partitions of a couple (B, s). Then the common refinement of these partitions is defined as the system $\mathcal{A} \vee \mathcal{B} = \{a_i \wedge b_j : a_i \in \mathcal{A}, b_j \in \mathcal{B}$, where $i = 1, 2, \dots, n;\ j = 1, 2, \dots, m\}$.

It may be noted that $\mathcal{A} \vee \mathcal{B}$ is also a partition of (B, s): Let $c_{ij} = \{a_i \wedge b_j : a_i \in \mathcal{A}, b_j \in \mathcal{B}$, where $i = 1, 2, \dots, n;\ j = 1, 2, \dots, m\}$, using monotonicity and $\vee$-orthogonality of $\mathcal{A}$ and $\mathcal{B}$, we can easily show that $\mathcal{A} \vee \mathcal{B}$ is a $\vee$-orthogonal system. And from (3.1), we have $s(\bigvee_{i,j=1}^{n,m}(a_i \wedge b_j)) = s(\bigvee_{i=1}^{n}(\bigvee_{j=1}^{m}(a_i \wedge b_j))) = \sum_{i=1}^{n} s(\bigvee_{j=1}^{m}(a_i \wedge b_j)) = \sum_{i=1}^{n} s(a_i) = 1$. So we get that the common refinement of two partitions of a couple (B, s) is also a partition of (B, s).

If $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$ is a partition of (B, s), then since $s(\bigvee_{i=1}^{n} a_i) = \sum_{i=1}^{n} s(a_i) = 1$, there exists at least one non-zero element in $\mathcal{A}$ with $s(a_i) > 0$.

Definition 3.2 Let the system $\mathcal{A} = \{a_1, a_2, \dots, a_n\} (n \in \mathbb{N})$ be a partition of a couple (B, s). Then the entropy H_s of $\mathcal{A}$ with respect to s is defined by

$$H_s(\mathcal{A}) = -\sum_{i=1}^{n} f(s(a_i)),$$

where the convex function $f : [0, \infty] \to \mathbb{R}$ is the Shannon's function given by $f(x) = x \log x$, if $x > 0$ and $f(0) = 0$.

For any $x, y \in [0, \infty]$, $f(xy) = xf(y) + yf(x)$. Since $f(x)$ is a convex, we have the following Jensen's inequality

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \le \sum_{i=1}^{n} \alpha_i f(x_i), \tag{3.2}$$

where α_i, $x_i \in [0,1]$ and $\sum_{i=1}^{n} \alpha_i = 1$. It may also be observed that $H_s(\mathcal{A}) \ge 0$.

Proposition 3.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be partitions of a couple (B, s). Then $H_s(\mathcal{A} \vee \mathcal{B}) \le H_s(\mathcal{A}) + H_s(\mathcal{B})$.*

Proof Assume that $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$ and $\mathcal{B} = \{b_1, b_2, \dots, b_m\}$ be partitions of (B, s). For a given i $(i = 1, 2, \dots, n)$, put $\alpha_j = s(b_j)$ and $x_j = s(a_i|b_j)$ $(j = 1, 2, \dots, m)$. Then $\alpha_j, x_j \in [0, 1]$, $\sum_{j=1}^{m} \alpha_j = \sum_{j=1}^{m} s(b_j) = 1$. Now using (2.1), we get

$$\sum_{j=1}^{m} \alpha_j x_j = \sum_{j=1}^{m} s(a_i \wedge b_j) = s\left(a_i \wedge \left(\bigvee_{j=1}^{m} b_j\right)\right) = s(a_i).$$

Also

$$\sum_{j=1}^{m}\alpha_j f(x_j) = \sum_{j=1}^{m} s(b_j)\frac{s(a_i\wedge b_j)}{s(b_j)}\log\frac{s(a_i\wedge b_j)}{s(b_j)}$$
$$= \sum_{j=1}^{m} s(a_i\wedge b_j)\log s(a_i\wedge b_j) - \sum_{j=1}^{m} s(a_i\wedge b_j)\log s(b_j)$$
$$= \sum_{j=1}^{m} f(s(a_i\wedge b_j)) - \sum_{j=1}^{m} s(a_i\wedge b_j)\log s(b_j).$$

Hence, by (3.2),

$$f(s(a_i)) \le \sum_{j=1}^{m} f(s(a_i\wedge b_j)) - \sum_{j=1}^{m} s(a_i\wedge b_j)\log s(b_j).$$

Thus

$$\sum_{i=1}^{n} f(s(a_i)) \le \sum_{i=1}^{n}\sum_{j=1}^{m} f(s(a_i\wedge b_j)) - \sum_{i=1}^{n}\sum_{j=1}^{m} s(a_i\wedge b_j)\log s(b_j).$$

From (3.1), we get $\sum_{i=1}^{n} s(a_i\wedge b_j) = s(b_j)$, $j = 1, 2, \ldots, m$. Therefore

$$\sum_{i=1}^{n}\sum_{j=1}^{m} s(a_i\wedge b_j)\log s(b_j) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} s(a_i\wedge b_j)\right)\log s(b_j) = \sum_{j=1}^{m} f(s(b_j)).$$

Hence

$$\sum_{i=1}^{n} f(s(a_i)) \le \sum_{i=1}^{n}\sum_{j=1}^{m} f(s(a_i\wedge b_j)) - \sum_{j=1}^{m} f(s(b_j)).$$

Therefore

$$H_s(\mathcal{A}\vee\mathcal{B}) \le H_s(\mathcal{A}) + H_s(\mathcal{B}).$$

□

Definition 3.3 Let the systems $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ be partitions of a couple (B, s). Then the conditional entropy $H_s(\mathcal{A}|\mathcal{B})$ is defined by

$$H_s(\mathcal{A}|\mathcal{B}) = -\sum_{j=1}^{m}\sum_{i=1}^{n} s(b_j) f(s(a_i|b_j)).$$

It may be observed that $H_s(\mathcal{A}|\mathcal{B}) \ge 0$.

Proposition 3.2 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be partitions of a couple (B, s). Then $H_s(\mathcal{A}\vee\mathcal{B}|\mathcal{C}) = H_s(\mathcal{A}|\mathcal{C}) + H_s(\mathcal{B}|\mathcal{A}\vee\mathcal{C})$.*

Proof Assume that $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$, $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ and $\mathcal{C} = \{c_1, c_2, \ldots, c_l\}$ are partitions of (B, s). If $s(a_i\wedge c_k) > 0$, where $i = 1, 2, \ldots, n$; $k = 1, 2, \ldots, l$, then for any j,

$j = 1, 2, \ldots, m$,

$$\begin{aligned} s(a_i \wedge b_j|c_k) &= \frac{s(a_i \wedge b_j \wedge c_k)}{s(c_k)} \\ &= \frac{s(a_i \wedge b_j \wedge c_k)s(a_i \wedge c_k)}{s(a_i \wedge c_k)s(c_k)} \\ &= s(b_j|a_i \wedge c_k)s(a_i|c_k), \\ H_s(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) &= -\sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n} s(c_k) f(s(a_i \wedge b_j|c_k)) \\ &= -\sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n} s(c_k) f(s(b_j|a_i \wedge c_k)s(a_i|c_k)) \\ &= -\sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n} s(c_k)[s(b_j|a_i \wedge c_k) f(s(a_i|c_k)) \\ &\quad + s(a_i|c_k) f(s(b_j|a_i \wedge c_k))] \\ &= -\sum_{k=1}^{l}\sum_{i=1}^{n} s(c_k) \sum_{j=1}^{m} s(b_j|a_i \wedge c_k) f(s(a_i|c_k)) \\ &\quad - \sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n} s(a_i \wedge c_k) f(s(b_j|a_i \wedge c_k)). \end{aligned}$$

But by (3.1), we obtain $\sum_{j=1}^{m} s(b_j|a_i \wedge c_k) = 1$. Thus

$$\begin{aligned} H_s(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) &= -\sum_{k=1}^{l}\sum_{i=1}^{n} s(c_k) f(s(a_i|c_k)) \\ &\quad - \sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n} s(a_i \wedge c_k) f(s(b_j|a_i \wedge c_k)) \\ &= H_s(\mathcal{A}|\mathcal{C}) + H_s(\mathcal{B}|\mathcal{A} \vee \mathcal{C}). \end{aligned}$$ □

Proposition 3.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be partitions of a couple (B, s). Then $H_s(\mathcal{A} \vee \mathcal{B}) = H_s(\mathcal{A}) + H_s(\mathcal{B}|\mathcal{A})$. Consequently, $H_s(\mathcal{A} \vee \mathcal{B}) \geq \max\{H_s(\mathcal{A}), H_s(\mathcal{B})\}$.*

Proof Assume that $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ are partitions of (B, s). Then

$$\begin{aligned} H_s(\mathcal{B}|\mathcal{A}) &= -\sum_{j=1}^{m}\sum_{i=1}^{n} s(a_i) f(s(b_j|a_i)) \\ &= -\sum_{j=1}^{m}\sum_{i=1}^{n} s(a_i) f\left(\frac{s(a_i \wedge b_j)}{s(a_i)}\right) \end{aligned}$$

$$= -\sum_{j=1}^{m}\sum_{i=1}^{n} s(a_i \wedge b_j)[\log s(a_i \wedge b_j) - \log s(a_i)]$$

$$= -\sum_{j=1}^{m}\sum_{i=1}^{n} s(a_i \wedge b_j)\log s(a_i \wedge b_j) + \sum_{i=1}^{n}\left[\sum_{j=1}^{m} s(a_i \wedge b_j)\right]\log s(a_i).$$

But by (3.1), we have $\sum_{j=1}^{m} s(a_i \wedge b_j) = s(a_i)$. Thus

$$H_s(\mathcal{B}|\mathcal{A}) = -\sum_{j=1}^{m}\sum_{i=1}^{n} s(a_i \wedge b_j)\log s(a_i \wedge b_j) + \sum_{i=1}^{n} s(a_i)\log s(a_i)$$
$$= H_s(\mathcal{A} \vee \mathcal{B}) - H_s(\mathcal{A}),$$

and so $H_s(\mathcal{A} \vee \mathcal{B}) = H_s(\mathcal{A}) + H_s(\mathcal{B}|\mathcal{A})$. □

Proposition 3.4 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be partitions of a couple (B, s). Then $H_s(\mathcal{A}|\mathcal{B} \vee \mathcal{C}) \le H_s(\mathcal{A}|\mathcal{B})$.*

Proof Assume that $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$, $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ and $\mathcal{C} = \{c_1, c_2, \ldots, c_l\}$ are partitions of (B, s). Symbolize $(b_j \wedge c_k)$ by e_{jk}; here $j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, l$. Then by (3.1),

$$\sum_{k=1}^{l} s(a_i \wedge e_{jk}) = \sum_{k=1}^{l} s(a_i \wedge b_j \wedge c_k) = s(a_i \wedge b_j).$$

Hence, for $s(e_{jk}) > 0$,

$$H_s(\mathcal{A}|\mathcal{B}) = -\sum_{j=1}^{m}\sum_{i=1}^{n} s(b_j) f\left(\frac{s(a_i \wedge b_j)}{s(b_j)}\right)$$
$$= -\sum_{j=1}^{m}\sum_{i=1}^{n} s(b_j) f\left(\sum_{k=1}^{l} \frac{s(a_i \wedge e_{jk}) s(e_{jk})}{s(b_j) s(e_{jk})}\right).$$

In view of the inequality (3.2), for $\alpha_k = \frac{s(e_{jk})}{s(b_j)}$ and $x_k = \frac{s(a_i \wedge e_{jk})}{s(e_{jk})}$, we get

$$H_s(\mathcal{A}|\mathcal{B}) \ge -\sum_{j=1}^{m}\sum_{i=1}^{n} s(b_j) \sum_{k=1}^{l} \frac{s(e_{jk})}{s(b_j)} f\left(\frac{s(a_i \wedge e_{jk})}{s(e_{jk})}\right)$$
$$= -\sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n} s(e_{jk}) f(s(a_i|e_{jk})).$$

Thus $H_s(\mathcal{A}|\mathcal{B} \vee \mathcal{C}) \le H_s(\mathcal{A}|\mathcal{B})$. □

Proposition 3.5 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be partitions of a couple (B, s). Then $H_s(\mathcal{A}|\mathcal{B}) + H_s(\mathcal{B}|\mathcal{C}) \ge H_s(\mathcal{A}|\mathcal{C})$.*

Proof Assume that $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$, $\mathcal{B} = \{b_1, b_2, \dots, b_m\}$ and $\mathcal{C} = \{c_1, c_2, \dots, c_l\}$ are partitions of (B, s). Then by Propositions 3.3 and 3.4, we obtain

$$\begin{aligned} H_s(\mathcal{A}|\mathcal{B}) + H_s(\mathcal{B}|\mathcal{C}) &= H_s(\mathcal{A} \vee \mathcal{B}) + H_s(\mathcal{B} \vee \mathcal{C}) - H_s(\mathcal{B}) - H_s(\mathcal{C}) \\ &= H_s(\mathcal{A} \vee \mathcal{B}) + H_s(\mathcal{C}|\mathcal{B}) - H_s(\mathcal{C}) \\ &\geq H_s(\mathcal{A} \vee \mathcal{B}) + H_s(\mathcal{C}|\mathcal{A} \vee \mathcal{B}) - H_s(\mathcal{C}) \\ &= H_s(\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) - H_s(\mathcal{C}) \\ &\geq H_s(\mathcal{A} \vee \mathcal{C}) - H_s(\mathcal{C}) = H_s(\mathcal{A}|\mathcal{C}). \end{aligned}$$ □

Proposition 3.6 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be partitions of a couple (B, s). Then $H_s(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) \leq H_s(\mathcal{A}|\mathcal{C}) + H_s(\mathcal{B}|\mathcal{C})$.*

Proof Follows from Propositions 3.2 and 3.4. □

4 s-Refinement and the Rokhlin Metric

Theorem 4.1 *Let (B, s) be a couple, where B is a Boolean algebra and s is a state on B. For partitions $\mathcal{A}$ and $\mathcal{B}$ of (B, s),*

$$d(\mathcal{A}, \mathcal{B}) = H_s(\mathcal{A}|\mathcal{B}) + H_s(\mathcal{B}|\mathcal{A})$$

defines a pseudo-metric on the family of all partitions of (B, s).

Proof As a consequence of the definition, it follows that $d(\mathcal{A}, \mathcal{B}) \geq 0$ and $d(\mathcal{A}, \mathcal{B}) = d(\mathcal{B}, \mathcal{A})$. Also $d(\mathcal{A}, \mathcal{A}) = H_s(\mathcal{A}|\mathcal{A}) = 0$. Finally, for partitions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ of (B, s), we obtain from Proposition 3.5 that

$$\begin{aligned} d(\mathcal{A}, \mathcal{C}) &= H_s(\mathcal{A}|\mathcal{C}) + H_s(\mathcal{C}|\mathcal{A}) \\ &\leq H_s(\mathcal{A}|\mathcal{B}) + H_s(\mathcal{B}|\mathcal{C}) + H_s(\mathcal{C}|\mathcal{B}) + H_s(\mathcal{B}|\mathcal{A}) \\ &= d(\mathcal{A}, \mathcal{B}) + d(\mathcal{B}, \mathcal{C}). \end{aligned}$$ □

Definition 4.1 Let $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$ and $\mathcal{B} = \{b_1, b_2, \dots, b_m\}$ be partitions of a couple (B, s), where B is a Boolean algebra and s is a state on B. Then $\mathcal{B}$ is called an s-refinement of $\mathcal{A}$, written as $\mathcal{A} \leq_s \mathcal{B}$ if, for each $b_j \in \mathcal{B}$, $j = 1, 2, \dots, m$, there exists $a_i \in \mathcal{A}$, $i = 1, 2, \dots, n$, such that $s(b_j \wedge a_i) = s(b_j)$.

Theorem 4.2 *For partitions $\mathcal{A}$ and $\mathcal{B}$ of a couple (B, s), $H_s(\mathcal{A}|\mathcal{B}) = 0$ if and only if $\mathcal{A} \leq_s \mathcal{B}$.*

Proof Let $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$ and $\mathcal{B} = \{b_1, b_2, \dots, b_m\}$ be partitions of (B, s) and $\mathcal{A} \leq_s \mathcal{B}$. Then, for each $b_j \in \mathcal{B}$, there exists $a_i \in \mathcal{A}$, where $i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$, such that $s(b_j \wedge a_i) = s(b_j)$. Consequently, $f(s(a_i|b_j)) = 0$ and so $H_s(\mathcal{A}|\mathcal{B}) = 0$. Conversely, if $H_s(\mathcal{A}|\mathcal{B}) = 0$, then we obtain that $f(s(a_i|b_j)) = 0$ for every i and j ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$). Hence either $s(a_i|b_j) = 0$ or it is 1. If $s(a_i|b_j) = 1$, then $s(b_j \wedge a_i) = s(b_j)$. Now, let $s(a_i|b_j) = 0$. By (3.1), for $b_j \in \mathcal{B}$,

$$\sum_{i=1}^{n} s(a_i \wedge b_j) = s(b_j).$$

If possible, let us assume that there is an element a_{i_0} such that $0 < s(a_{i_0} \wedge b_j) < s(b_j)$. Then $s(b_j)f(s(a_{i_0}|b_j)) \neq 0$, which contradicts the hypothesis that $H_s(\mathcal{A}|\mathcal{B}) = 0$. Hence we deduce that there exists an i_p, $1 \leq i_p \leq n$, such that $s(b_j \wedge a_{i_p}) = s(b_j)$. Thus $\mathcal{A} \leq_s \mathcal{B}$. □

Proposition 4.1 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be partitions of a couple (B, s). Then $\mathcal{A} \leq_s \mathcal{B}$ and $\mathcal{B} \leq_s \mathcal{C}$ imply that $\mathcal{A} \leq_s \mathcal{C}$.*

Proof Assume that $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$, $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ and $\mathcal{C} = \{c_1, c_2, \ldots, c_l\}$ be partitions of (B, s). Since $\mathcal{A} \leq_s \mathcal{B}$, then for each $b_j \in \mathcal{B}$, there exists $a_i \in \mathcal{A}$ (where $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$) such that $s(b_j \wedge a_i) = s(b_j)$ and so from the modularity of state s, we have $s(b_j \vee a_i) = s(a_i)$. And also $\mathcal{B} \leq_s \mathcal{C}$ which implies that for each $c_k \in \mathcal{C}$, there exists $b_j \in \mathcal{B}$ (where $k = 1, 2, \ldots, m$; $j = 1, 2, \ldots, m$) such that $s(c_k \wedge b_j) = s(c_k)$, and so $s(c_k \vee b_j) = s(b_j)$. Now we have

$$\begin{aligned}
s(c_k) &= s(c_k \wedge b_j) \\
&= s(c_k \wedge b_j) + s(a_i) - s(a_i) \\
&= s((c_k \wedge b_j) \vee a_i) + s((c_k \wedge b_j) \wedge a_i) - s(a_i) \\
&= s((c_k \vee a_i) \wedge (b_j \vee a_i)) + s(c_k \wedge b_j \wedge a_i) - s(a_i) \\
&= s(c_k \vee a_i) + s(b_j \vee a_i) - s((c_k \vee a_i) \vee (b_j \vee a_i)) \\
&\quad + s(c_k \wedge b_j \wedge a_i) - s(a_i) \\
&= s(c_k \vee a_i) - s(c_k \vee b_j \vee a_i) + s(c_k \wedge b_j \wedge a_i) \\
&\leq s(c_k \wedge b_j \wedge a_i) \leq s(c_k \wedge a_i).
\end{aligned}$$

Thus $s(c_k) = s(c_k \wedge a_i)$. Hence $\mathcal{A} \leq_s \mathcal{C}$. □

Remark 4.1 Let $\mathfrak{P}_s$ denote the family of all partitions of a couple (B, s), where B is a Boolean algebra and s is a state on B. For $\mathcal{A}$ and $\mathcal{B} \in \mathfrak{P}_s$, define a relation $\sim$ as follows:

$$\mathcal{A} \sim \mathcal{B} \quad \Longleftrightarrow \quad \mathcal{A} \leq_s \mathcal{B} \quad \text{and} \quad \mathcal{B} \leq_s \mathcal{A}.$$

In view of Theorem 4.2, $\sim$ is an equivalence relation on $\mathfrak{P}_s$, and then the pseudo-metric d as defined in Theorem 4.1, turns out to be a metric on $\mathfrak{P}_s/_\sim$. Following the terminology of the classical case, we call this metric the *Rokhlin metric* (cf. [12, 25]). Thus we have the following:

Theorem 4.3 *For $\mathcal{A}, \mathcal{B} \in \mathfrak{P}_s/_\sim$, $d(\mathcal{A}, \mathcal{B}) = H_s(\mathcal{A}|\mathcal{B}) + H_s(\mathcal{B}|\mathcal{A})$ is a metric on $\mathfrak{P}_s/_\sim$.*

5 Partition of Quantum Spaces

We now extend the theory developed in the previous Sects. 3 and 4 to a quantum space (L, s), where L is an orthomodular lattice and s is a Bayessian state on L (i.e. s has the Bayes' property).

For a (finite) system $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ of elements of L the definitions of $\vee$-orthogonal system and a partition of the quantum space (L, s), where L is an orthomodular lattice and s is a state on L, continue to be valid. If $\mathcal{A}$ is a $\vee$-orthogonal system on L, then it is straightforward to see that

$$s\left(\bigvee_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} s(a_i).$$

The *common refinement* of two partitions $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ of (L, s) may also be defined as in the case of Boolean algebras (Definition 3.1):

$$\mathcal{A} \vee \mathcal{B} := \{a_i \wedge b_j : a_i \in \mathcal{A},\ b_j \in \mathcal{B},\ i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m\}.$$

The common refinement $\mathcal{A} \vee \mathcal{B}$ of partitions $\mathcal{A}$ and $\mathcal{B}$ turns out to be a partition of (L, s), provided s has the *Bayes' property*:

$$s\left(\bigvee_{j=1}^{m} (a \wedge b_j)\right) = s(a), \quad a \in L$$

(see [29]). But in this case (i.e. when s is a Bayessian state on L), s annihilates all (upper) commutators in L, i.e.

$$s(\overline{com}(a, b)) = 0, \quad \forall a, b \in L,$$

where $\overline{com}(a, b) := (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b'), a, b \in L$, which according to [19, Chap. 5], is equivalent to the existence of joint distribution $x_a, x_b \in L$, which is further equivalent to the *Bell's third inequality*:

$$s(a) + s(b) + s(c) - s(a \wedge b) - s(b \wedge c) - s(a \wedge c) \le 1, \quad a, b, c \in L, \tag{5.1}$$

from [21]. Inequalities (5.1) are satisfied on a quantum space (L, s) if and only if (L, s) is equivalent, from the point of view of probability theory, to a couple (B, s_0), where B is a Boolean algebra and s_0 is a state on B.

Alternately, if we consider the quantum space (L, s), where L is an OML and s is a state on L satisfying the Bayes' property, then $s(\overline{com}(a, b)) = 0$, for all $a, b \in L$. Therefore $s/J_c = 0$, i.e. the state s vanishes on the Marsden's ideal J_c ([16] and Theorem 5 in [21]) and hence $B := L/J_c$ (the quotient of L corresponding to J_c) is a Boolean algebra. We can now introduce a state s_0 on B by $s_0[a] = s(a), a \in L$, so that $s_0 \circ \phi = s$, where $\phi : L \to B$ is the natural homomorphism (see [21]), and in this way we can transfer everything to Boolean algebras. Thus we can replace the quantum space (L, s) (where s is a Bayessian state on L) equivalently by the couple (B, s_0).

Further theory on commutators and the Bell inequalities may be seen in [2, 16, 17, 19–22].

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